Math 4650
Homework 3
Solutions

1

$$\sum_{n=2}^{\infty}\left(\frac{2}{7}\right)^n = \left(\frac{2}{7}\right)^2 + \left(\frac{2}{7}\right)^3 + \left(\frac{2}{7}\right)^4 + \cdots$$

$$= \left(\frac{2}{7}\right)^2\left[1 + \left(\frac{2}{7}\right) + \left(\frac{2}{7}\right)^2 + \cdots\right]$$

$$= \left(\frac{2}{7}\right)\left[\frac{1}{1-\frac{2}{7}}\right] = \frac{2}{7}\left[\frac{7}{5}\right] = \frac{2}{5}$$

$$\boxed{\begin{array}{l} 1 + x + x^2 + \cdots = \frac{1}{1-x} \\ \text{if } -1 < x < 1 \end{array}}$$

2

$$\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{2n} = \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^4 + \left(\frac{1}{3}\right)^6 + \left(\frac{1}{3}\right)^8 + \cdots$$

$$= \left(\frac{1}{3^2}\right) + \left(\frac{1}{3^2}\right)^2 + \left(\frac{1}{3^2}\right)^3 + \left(\frac{1}{3^2}\right)^4 + \cdots$$

$$= \frac{1}{9} + \left(\frac{1}{9}\right)^2 + \left(\frac{1}{9}\right)^3 + \left(\frac{1}{9}\right)^4 + \cdots$$

$$= \frac{1}{9}\left[1 + \frac{1}{9} + \frac{1}{9^2} + \frac{1}{9^3} + \cdots\right]$$

$$= \frac{1}{9}\left[\frac{1}{1 - 1/9}\right] = \frac{1}{9}\left[\frac{9}{8}\right] = \frac{1}{8}$$

$$\boxed{1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \text{ if } -1 < x < 1}$$

3

Use partial fractions:

$$\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$$

multiply by $(n+1)(n+2)$ on both sides

$$1 = A(n+2) + B(n+1)$$

$\underline{n=-2:}$ $1 = A(-2+2) + B(-2+1)$

$1 = 0 - B$

$-1 = B$

$\underline{n=-1:}$ $1 = A(-1+2) + B(-1+1)$

$1 = A(1) + 0$

$1 = A$

So, $$\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

Let s_k be the partial sum sequence.

Then,

$$s_1 = \underbrace{\left(\frac{1}{2} - \frac{1}{3}\right)}_{n=1 \text{ term}}$$

$$s_2 = \underbrace{\left(\frac{1}{2} - \frac{1}{3}\right)}_{n=1 \text{ term}} + \underbrace{\left(\frac{1}{3} - \frac{1}{4}\right)}_{n=2 \text{ term}} = \frac{1}{2} - \frac{1}{4}$$

$$s_3 = \underbrace{\left(\frac{1}{2} - \frac{1}{3}\right)}_{n=1 \text{ term}} + \underbrace{\left(\frac{1}{3} - \frac{1}{4}\right)}_{n=2 \text{ term}} + \underbrace{\left(\frac{1}{4} - \frac{1}{5}\right)}_{n=3 \text{ term}} = \frac{1}{2} - \frac{1}{5}$$

$$\vdots \qquad \vdots$$

In general,

$$s_k = \frac{1}{2} - \frac{1}{k+2}$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \lim_{k\to\infty} s_k = \lim_{k\to\infty} \left(\frac{1}{2} - \frac{1}{k+2}\right)$$

$$= \frac{1}{2} - 0$$

$$= \frac{1}{2}$$

4

Use partial fractions:

$$\frac{2}{n^2-1} = \frac{2}{(n+1)(n-1)} = \frac{A}{n+1} + \frac{B}{n-1}$$

multiply by $(n+1)(n-1)$ on both sides

$$2 = A(n-1) + B(n+1)$$

$\underline{n=1:}$ $2 = A(1-1) + B(1+1)$

$2 = 0 + 2B$

$1 = B$

$\underline{n=-1:}$ $2 = A(-1-1) + B(-1+1)$

$2 = A(-2) + 0$

$-1 = A$

Thus,

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \sum_{n=2}^{\infty} \left(\frac{-1}{n+1} + \frac{1}{n-1} \right)$$

Let s_k denote the partial sum sequence.

Then,

$$s_2 = \underbrace{\left(\frac{-1}{2+1} + \frac{1}{2-1} \right)}_{n=2 \text{ term}} = \frac{-1}{3} + 1 = \boxed{\frac{2}{3}}$$

$$S_3 = \underbrace{\left(-\frac{1}{3}+1\right)}_{n=2 \text{ term}} + \underbrace{\left(-\frac{1}{4}+\frac{1}{2}\right)}_{n=3 \text{ term}} = \boxed{\frac{3}{2}-\frac{1}{3}-\frac{1}{4}}$$

$$S_4 = \underbrace{\left(-\cancel{\frac{1}{3}}+1\right)}_{n=2 \text{ term}} + \underbrace{\left(-\frac{1}{4}+\frac{1}{2}\right)}_{n=3 \text{ term}} + \underbrace{\left(-\frac{1}{5}+\cancel{\frac{1}{3}}\right)}_{n=4 \text{ term}}$$

$$= 1+\frac{1}{2}-\frac{1}{4}-\frac{1}{5} = \boxed{\frac{3}{2}-\frac{1}{4}-\frac{1}{5}}$$

$$S_5 = S_4 + \underbrace{\left(-\frac{1}{6}+\frac{1}{4}\right)}_{n=5 \text{ term}}$$

$$= \left(\frac{3}{2}-\cancel{\frac{1}{4}}-\frac{1}{5}\right) + \left(-\frac{1}{6}+\cancel{\frac{1}{4}}\right)$$

$$= \boxed{\frac{3}{2}-\frac{1}{5}-\frac{1}{6}}$$

$$S_6 = S_5 + \underbrace{\left(-\frac{1}{7}+\frac{1}{5}\right)}_{n=6 \text{ term}}$$

$$= \left(\frac{3}{2}-\cancel{\frac{1}{5}}-\frac{1}{6}\right) + \left(-\frac{1}{7}+\cancel{\frac{1}{5}}\right)$$

$$= \boxed{\frac{3}{2}-\frac{1}{6}-\frac{1}{7}}$$

$$\vdots \qquad \vdots$$

We see that in general

$$S_k = \frac{3}{2} - \frac{1}{k} - \frac{1}{k+1}$$

Thus,

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \lim_{k\to\infty} S_k$$

$$= \lim_{k\to\infty} \left(\frac{3}{2} - \frac{1}{k} - \frac{1}{k+1} \right)$$

$$= \frac{3}{2} - 0 - 0$$

$$= \frac{3}{2}$$

$\boxed{⑤(a)}$

Suppose that $\sum_{n=1}^{\infty} a_n = A$ and $\alpha \in \mathbb{R}$.

Let $s_k = a_1 + a_2 + \dots + a_k$

be the partial sums for $\sum_{n=1}^{\infty} a_n$.

Since $\sum_{n=1}^{\infty} a_n = A$ we know $\lim_{k\to\infty} s_k = A$.

Note that the partial sums of $\sum_{n=1}^{\infty} \alpha a_n$ are

$$\alpha a_1 + \alpha a_2 + \dots + \alpha a_k = \alpha(a_1 + a_2 + \dots + a_k)$$
$$= \alpha s_k$$

Then,

$$\sum_{n=1}^{\infty} \alpha a_k = \lim_{k\to\infty} \alpha s_k = \alpha \lim_{k\to\infty} s_k = \alpha A.$$

algebra of sequence limits

$\blacksquare$

5 (b)

Suppose that $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$.

Let $s_k = a_1 + a_2 + \dots + a_k$

and $t_k = b_1 + b_2 + \dots + b_k$

be the partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$.

Then, $\lim_{k\to\infty} s_k = A$ and $\lim_{k\to\infty} t_k = B$.

The partial sums of $\sum_{n=1}^{\infty}(a_n + b_n)$ are

$$(a_1+b_1)+(a_2+b_2)+\dots+(a_k+b_k)$$

$$= (a_1+a_2+\dots+a_k)+(b_1+b_2+\dots+b_k)$$

$$= s_k + t_k$$

Thus,

$$\sum_{n=1}^{\infty}(a_n+b_n) = \lim_{k\to\infty}(s_k+t_k) = A+B$$

algebra of sequence limits

6

($\Rightarrow$) Suppose that (a_n) converges to 0.

Let $\varepsilon > 0$.

Then there exists $N > 0$ where if $n \geq N$
then $|a_n - 0| < \varepsilon$.

So if $n \geq N$ then $|a_n| < \varepsilon$

So if $n \geq N$ then $||a_n|| < \varepsilon$

So if $n \geq N$ then $||a_n| - 0| < \varepsilon$

Thus, $(|a_n|)$ converges to 0.

($\Leftarrow$) Suppose $(|a_n|)$ converges to 0.

Then there exists $N > 0$ where if $n \geq N$
then $||a_n| - 0| < \varepsilon$.

Thus if $n \geq N$ then $||a_n|| < \varepsilon$

So if $n \geq N$ then $|a_n| < \varepsilon$.

So if $n \geq N$ then $|a_n - 0| < \varepsilon$

Thus, (a_n) converges to 0. $\boxed{//}$

7 (a)

$$\lim_{n\to\infty} \frac{2n}{3n+1} = \lim_{n\to\infty} \frac{2}{3+1/n} = \frac{2}{3+0} = \frac{2}{3}$$

divide top/bottom by n

Since $\lim_{n\to\infty} \frac{2n}{3n+1} \neq 0$, by the divergence test, we know $\sum_{n=1}^{\infty} \frac{2n}{3n+1}$ diverges.

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7 (b)

Note that $\frac{1}{n^3+n^2+n+1} \leq \frac{1}{n^3+n^3+n^3+n^3} = \frac{4}{n^3}$

$n \geq 1$

We know that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges ($p=3$ series)

Thus from HW problem 5 above $\sum_{n=1}^{\infty} \frac{4}{n^3}$ converges.

By the comparison test we know that $\sum_{n=1}^{\infty} \frac{1}{n^3+n^2+n+1}$ converges

7 (c)

Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+1} = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$

where $a_n = \frac{1}{n^2+1}$ are positive real numbers.

We use the alternating series test to show this series converges.

We have that

$$a_{n+1} = \frac{1}{(n+1)^2+1} = \frac{1}{n^2+2n+2} < \frac{1}{n^2+1} = a_n$$

$$n^2+2n+2 > n^2+0+1 = n^2+1$$

So, the sequence (a_n) is monotonically decreasing.

Also, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2+1} = 0$.

Thus, by the alternating series test

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+1}$ converges.

$\boxed{⑦(d)}$

$$\lim_{n\to\infty}\left|(-1)^n \frac{n^3}{2n^3+1}\right| = \lim_{n\to\infty} \underbrace{\left|(-1)^n\right|}_{1}\cdot\left|\frac{n^3}{2n^3+1}\right|$$

$$= \lim_{n\to\infty} \frac{n^3}{2n^3+1} = \lim_{n\to\infty} \frac{1}{2+1/n^3} = \frac{1}{2+0} = \frac{1}{2}$$

$\frac{n^3}{2n^3+1} > 0$ since $n \geq 1$

By problem 6 since $\lim_{n\to\infty}\left|(-1)^n \frac{n^3}{2n^3+1}\right| \neq 0$

We know $\lim_{n\to\infty} (-1)^n \frac{n^3}{2n^3+1} \neq 0$.

Thus, by the divergence test

$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{2n^3+1}$ diverges.

7(e) Let s_k be the partial sums of

$\sum_{n=1}^{\infty}(\sqrt{n+1}-\sqrt{n})$. Then,

$$s_1 = \sqrt{2}-\sqrt{1} = \sqrt{2}-1$$
$$s_2 = (\sqrt{2}-\sqrt{1})+(\sqrt{3}-\sqrt{2}) = \sqrt{3}-1$$
$$s_3 = s_2+(\sqrt{4}-\sqrt{3}) = (\sqrt{3}-1)+(\sqrt{4}-\sqrt{3}) = \sqrt{4}-1$$
$$\vdots \qquad \vdots$$

In general, $s_k = \sqrt{k}-1$

We see that (s_k) is unbounded so

$\lim_{k\to\infty} s_k$ does not exist.

Thus, $\sum_{n=1}^{\infty}(\sqrt{n+1}-\sqrt{n})$ diverges

Let $M>0$.

Then, set $k=(M+2)^2$ to get $s_k = \sqrt{(M+2)^2}-1 = M+1 > M$

8 (a)

Let $s_k = |a_1| + |a_2| + \dots + |a_k|$

and $t_k = a_1 + a_2 + \dots + a_k$

be the partial sums of $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} a_n$

Since s_k converges, it is a Cauchy sequence

Let $\varepsilon > 0$.

Then there exists $N > 0$ where if $l \geq m \geq N$ then

$$|s_l - s_m| < \varepsilon.$$

That is if $l \geq m \geq N$, then

$$\left| (|a_1| + |a_2| + \dots + |a_l|) - (|a_1| + |a_2| + \dots + |a_m|) \right|$$

$$= \left| |a_{m+1}| + |a_{m+2}| + \dots + |a_l| \right| < \varepsilon$$

So if $l \geq m \geq N$, then $|a_{m+1}| + |a_{m+2}| + \dots + |a_l| < \varepsilon$

Then if $l \geq m \geq N$, we have

$$|t_l - t_m| = |a_{m+1} + a_{m+2} + \dots + a_l|$$

$$\leq |a_{m+1}| + |a_{m+2}| + \dots + |a_l| < \varepsilon$$

So, the sequence (t_k) is Cauchy also.

Thus, $\sum_{n=1}^{\infty} a_n$ converges.

8 (b)

The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges

but $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

from class

$\boxed{9}$

If $-1 < r < 1$, then $|r| < 1$.

We will show that $\lim |r^n| = 0$.

By problem 6, this will imply that $\lim_{n\to\infty} r^n = 0$

If $r = 0$, then $\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} 0 = 0$

If $r \neq 0$ and $|r| < 1$, then $\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0$

HW 3 problem 2

Thus, $\lim_{n\to\infty} |r^n| = 0$

By problem 6, we get $\lim_{n\to\infty} r^n = 0$

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